Existence Results for Second Order Boundary Value Problem of Impulsive Dynamic Equations on Time Scales

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Abstract

In this paper, Schaefer's fixed point theorem and nonlinear alternative of Leray-Schauder type are used to investigate the existence of solutions for second order boundary value problem for impulsive dynamic equations on time scales.

Key words and phrases: Impulsive dynamic equations, boundary value problem, delta derivative, fixed point, time scale, measure chains.
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1 Introduction

This paper is concerned with the existence of solutions of boundary value problems for impulsive dynamic equations on time scales. We consider the following boundary value problem,

$$-y^{\Delta\Delta}(t) = f(t, y(t)),\ t \in J := [0,1],\ t \neq t_k, \quad k = 1, \dots, m, \tag{1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y^{\Delta}(t_k^+) - y^{\Delta}(t_k) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{3}$$

$$y(0) = y(1) = 0, \tag{4}$$

where T is time scale, $[0,1] \subset \mathrm{T}$, $f : \mathrm{T} \times \mathbb{R} \to \mathbb{R}$, is a given function, $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in \mathrm{T}$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Impulsive differential equations have become important in recent years in mathematical models of real processes and they rise in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [12], Lakshmikantham *et al* [23], and Samoilenko and Perestyuk [25] and the references therein. In recent years differential equations on times scales have received much attention. We refer to the books by Agarwal and O'Regan [6], Bohner and Peterson [14], [15], Lakshmikantham *et al* [24] and the papers by Anderson [7], [9], Agarwal *et al* [1], [2], [3], [4], Bohner and Guseinov [13], Bohner and Eloe [17], Bohner and Peterson [16], Erbe and Peterson [19], [20]. The times scales calculus has tremendous potential for applications in mathematical models of real processes and phenomena, for example in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, see the monographs of Aulback and Hilger [11], Bohner and Peterson, [14], [15] Lakshmikantham *et al* [24], and to the references therein. The existence of solutions of boundary value problems on a measure chain (i.e. time scales) was recently studied by Agarwal and O'Regan [5], Anderson [8], [10], Henderson [21] and Sum and Li [27]. Very recently [22] Henderson investigate the existence of double solutions to an impulsive dynamic equation. The main theorems of this paper complement the very little existence results devoted to impulsive dynamic equations on a time scale. We shall prove our two existence results for the problem (1)-(4) by using Scheafer's fixed point theorem [26] and nonlinear alternative of Leray-Schauder type [18].

2 Preliminaries

We will briefly recall some basic definitions and facts from the times scales calculus that we will use in the sequel.

A time scale T is a closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathrm{T} \to \mathrm{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathrm{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathrm{T} : s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathrm{T}$ and $\sup \emptyset := \inf \mathrm{T}$) are well defined. The point $t \in \mathrm{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If T has a right-scattered minimum m, define $\mathrm{T}_k := \mathrm{T} - \{m\}$; otherwise, set $\mathrm{T}_k = \mathrm{T}$. If T has a left-scattered maximum M, define $\mathrm{T}^k := \mathrm{T} - \{M\}$; otherwise, set $\mathrm{T}^k = \mathrm{T}$. The notations $[a, b], [a, b)$, and so on, will denote time scales intervals

$$[a, b] = \{t \in \mathrm{T} : a \leq t \leq b\},$$

where $a, b \in \mathrm{T}$ with $a < \rho(b)$.

Definition 2.1 *Let X be a Banach space. The function $f : \mathrm{T} \to X$ is called rd−continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, write $f \in C_{rd}(\mathrm{T}) = C_{rd}(\mathrm{T}, X)$. Let $t \in \mathrm{T}^k$, the Δ derivative of f at t, denoted*

$f^{\Delta}(t)$, be the number (provided it exists) if for all $\varepsilon > 0$ there exists a neighboord U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$, at fix t. Let F be a function. It is called antiderivative of $f : \mathrm{T} \to X$ provided

$$F^{\Delta}(t) = f(t) \quad \textit{for each } t \in \mathrm{T}^k.$$

A function $p : \mathrm{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0 \ \text{ for all } \ t \in \mathrm{T},$$

where $\mu(t) = \sigma(t) - t$ which called the graininess function. $C([0,b], \mathbb{R})$ is the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ where $[a,b] \subset \mathrm{T}$ with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : t \in [a,b]\}.$$

Remark 2.1 *(i) If f is continuous, then f rd−continuous.*

(ii) If f is delta differentiable at t then f is continuous at t.

3 Main Results

We will assume for the remainder of the paper that, for each $k = 1, \ldots, m$, the points of impulse t_k are right dense. In order to define the solution of (1)–(4) we shall consider the following space

$$PC = \left\{ y : [0,1] \longrightarrow \mathbb{R} : \begin{array}{l} y_k \in C(J_k, \mathbb{R}), k = 0, \ldots, m \text{ and there exist } y(t_k^-) \\ \text{and } y(t_k^+), k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k) \end{array} \right\}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_{J_k}, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}] \subset [0,1]$, $k = 0, \ldots, m$. Let us start by defining what we mean by a solution of problem (1)-(4).

Definition 3.1 *A function $y \in PC \cap \cup_{k=0}^{m} C^2((t_k, t_{k+1}), \mathbb{R})$ is said a solution of (1)–(4) if it satisfies the differential equation*

$$-y^{\Delta\Delta}(t) = f(t, y(t)) \qquad \textit{everywhere on } J \backslash \{t_k\}, \ k = 1, \ldots, m,$$

and for each $k = 1, \ldots, m$ the function y satisfies the conditions $y(t_k^+) - y(t_k) = I_k(y(t_k^-))$, $y^{\Delta}(t_k^+) - y^{\Delta}(t_k) = \bar{I}_k(y(t_k^-))$ and the boundary conditions $y(0) = y(1) = 0$.

Lemma 3.1 *Let* $f : \mathrm{T} \to \mathbb{R}$ *be rd-continuous. If* y *is a solution of the equation*

$$y(t) = \int_0^1 G(t,s)f(s)\Delta s + \sum_{k=1}^{m} W_k(t, y(t_k)), \tag{5}$$

where

$$G(t,s) = \begin{cases} (1-t)\sigma(s), & \text{if } 0 \le s \le t \\ (1-\sigma(s))t, & \text{if } t \le s \le 1 \end{cases}$$

and

$$W_k(t, y(t_k)) = \begin{cases} t[-I_k(y(t_k) - (1-t_k)\bar{I}_k(y(t_k))], & \text{if } 0 \le t \le t_k \\ (1-t)[I_k(y(t_k)) - t_k\bar{I}_k(y(t_k))], & \text{if } t_k < t \le 1, \end{cases}$$

then y *is a solution of the boundary value problem*

$$-y^{\Delta\Delta}(t) = f(t), \tag{6}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{7}$$

$$y^{\Delta}(t_k^+) - y^{\Delta}(t_k) = \bar{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{8}$$

$$y(0) = y(1) = 0. \tag{9}$$

Proof. Let y satisfies the integral equation (5), then y is solution of the problem (6)–(9). Firstly $y(0) = y(1) = 0$. Let $t \in [0,1] \backslash \{t_1, \ldots, t_m\}$, then we have

$$y(t) = \int_0^1 G(t,s))f(s)\Delta s + \sum_{k=1}^{m} W_k(t, y(t_k)).$$

Hence

$$\begin{aligned} y^{\Delta}(t) &= \left[\int_0^1 G(t,s)f(s)\Delta s + \sum_{k=1}^{m} W_k(t_k, y(t_k))\right]^{\Delta} \\ &= \left[\int_0^1 G(t,s)f(s)\Delta s\right]^{\Delta} + \left[\sum_{k=1}^{m} W_k(t, y(t_k))\right]^{\Delta} \\ &= \left[\int_0^t (1-t)\sigma(s)f(s)\Delta s\right]^{\Delta} + \left[\int_t^1 (1-\sigma(s))tf(s)\Delta s\right]^{\Delta} + \sum_{k=1}^{m} W_k^{\Delta}(t,y) \\ &= -\int_0^t \sigma(s)f(s)\Delta s + \int_t^1 (1-\sigma(s))f(s)\Delta s + \sum_{k=1}^{m} W_k^{\Delta}(t,y), \end{aligned}$$

where

$$W_k^{\Delta}(t,y) = \begin{cases} -I_k(y(t_k)) - (1-t_k)\bar{I}_k(y(t_k))], & \text{if } 0 \le t \le t_k \\ -[I_k(y(t_k)) - t_k\bar{I}_k(y(t_k))), & \text{if } t_k < t \le 1, \end{cases}$$

and

$$W_k^{\Delta\Delta}(t,y) = 0 \quad \text{for } k = 1, \ldots, m.$$

Thus

$$\begin{aligned}
y^{\Delta\Delta}(t) &= \left[-\int_0^t \sigma(s)f(s)\Delta s + \int_t^1 (1-\sigma(s))f(s)\Delta s + \sum_{k=1}^m W_k^{\Delta}(t,y)\right]^{\Delta} \\
&= \left[-\int_0^t \sigma(s)f(s)\Delta s\right]^{\Delta} + \left[\int_t^1 (1-\sigma(s))f(s)\Delta s\right]^{\Delta} \\
&= f(t).
\end{aligned}$$

Clearly we have

$$y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \bar{I}_k(y(t_k)), \quad \text{for } k = 1, \dots, m.$$

From the definition of y we can prove that

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad \text{for } k = 1, \dots, m.$$

□

Let us introduce the following hypotheses, which are assumed hereafter:

(H1) The functions $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\sigma : \mathrm{T} \longrightarrow \mathrm{T}$ are continuous.

(H2) There exist constants $c_k, \bar{c}_k$ such that

$$|I_k(y)| \le c_k, \quad |\bar{I}_k(y)| \le \bar{c}_k \;\text{ for each } k = 1, \dots, m, \text{ and for all } y \in \mathbb{R}.$$

(H3) There exists a function $p \in C([0,1], \mathbb{R}_+)$ such that

$$\|f(t,y)\| \le p(t) \;\text{ for each } (t,y) \in [0,1] \times \mathbb{R}.$$

Theorem 3.1 *Suppose that hypotheses (H1)–(H3) are satisfied. Then the impulsive BVP (1)–(4) has at least one solution.*

Proof. Transform the BVP (1)–(4) into a fixed point problem. Consider the operator $N : PC \longrightarrow PC$ defined by:

$$(Ny)(t) = \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^m W_k(t, y(t_k)).$$

Remark 3.1 *Clearly from Lemma 3.1 the fixed points of N are solutions to (1)–(4).*

We shall show that N satisfies the assumptions of Schaefer's fixed point theorem. The proof will be given in several steps. We show first that N is continuous and completely continuous.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC. Then

$$\begin{aligned}|N(y_n)(t) - N(y)(t)| \;\leq\; & \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^1 |f(s,y_n(s)) - f(s,y(s))|\Delta s \\ & + \sum_{k=1}^{m} |W_k(t,y_n(t_k)) - W_k(t,y(t_k))|.\end{aligned}$$

Since $f, I_k, \bar{I}_k$ are continuous, we have

$$\begin{aligned}\|N(y_n) - N(y)(t)\|_{PC} \;\leq\; & \sup_{(t,s)\in J\times J}|G(t,s)|\|f(.,y_n(.)) - f(.,y(.))\|_\infty \\ & +2\sum_{k=1}^{m}[|I_k(y_n(t_k)) - I_k(y(t_k))| + |\bar{I}_k(y_n(t_k)) - \bar{I}_k(y(t_k))|].\end{aligned}$$

Thus

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad as \; n \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in PC.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in PC : \|y\|_{PC} \leq q\}$ one has $\|Ny\|_{PC} \leq \ell$. For each $t \in [0,1]$ we have

$$(Ny)(t) = \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^{m} W_k(t,y(t_k)).$$

From (H2), (H3) we have

$$\begin{aligned}|(Ny)(t)| &= \left|\int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^{m} W_k(t,y(t_k))\right| \\ &\leq \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^1 |f(s,y(s))|\Delta s + \sum_{k=0}^{m}|W_k(t,y(t_k))| \\ &\leq \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^1 p(s)\Delta s + 2\sum_{k=0}^{m}[c_k + \bar{c}_k].\end{aligned}$$

Thus

$$\|Ny\|_{PC} \leq p_* + 2\sum_{k=0}^{m}[c_k + \bar{c}_k] := \ell,$$

where

$$p_* = \sup_{(t,s)\in J\times J}|G(t,s)| \sup_{t\in[0,1]} p(t).$$

Step 3: *N maps bounded sets into equicontinuous sets of PC.*

Let $r_1, r_2 \in J,\ r_1 < r_2$ and B_q be a bounded set of PC as in Step 2. Let $y \in B_q$, then

$$\begin{aligned}
\|(Ny)(r_2) - (Ny)(r_1)\| &\leq \int_0^1 |G(r_2,s) - G(r_1,s)||f(s,y(s))|\Delta s \\
&\quad + \sum_{k=1}^{m} |W_k(r_2, y(t_k)) - W_k(r_1, y(t_k))| \\
&\leq \int_0^1 |G(r_2,s) - G(r_1,s)|p(s)\Delta s \\
&\quad + \sum_{k=1}^{m} |W_k(r_2, y(t_k)) - W_k(r_1, y(t_k))|.
\end{aligned}$$

The right-hand side tends to zero as $r_2 - r_1 \longrightarrow 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli Theorem we can conclude that $N : PC \longrightarrow PC$ is continuous and completely continuous.

Step 4: *Now it remains to show that the set*

$$\mathcal{E}(N) := \{y \in PC : y = \lambda N(y), \ \textit{for some} \ \ 0 < \lambda < 1\}$$

is bounded. As in step 3 we can prove that $\mathcal{E}(N)$ is bounded.

Set $X := PC$. As a consequence of Schaefer's fixed point theorem theorem ([26] p. 29) we deduce that G has a fixed point y which is a solution to problem (1)–(4). □

We present now a result for the problem (1)-(4) in the spirit of the nonlinear alternative of Leray-Schauder type [18].

Theorem 3.2 *Suppose that hypotheses (H1)-(H2) and the following condition are satisfied:*

(A1) There exists a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$, $\bar{p} \in C([0,1], \mathbb{R}_+)$ *and nonnegative number* $r > 0$ *such that*

$$\|F(t,y)\| \leq \bar{p}(t)\psi(|y|) \ \ \textit{for each} \ y \in \mathbb{R}$$

and

$$\frac{r}{\sup\limits_{(t,s)\in[0,1]\times[0,1]} \psi(r) \displaystyle\int_0^1 \bar{p}(s)\Delta s + 2\sum_{k=0}^{m}[c_k + \bar{c}_k]} > 1.$$

Then the impulsive IVP (1)–(4) has at least one solution.

Proof. Transform the BVP (1)–(4) into a fixed point problem. Consider the operator N defined in the proof of Theorem 3.1. We shall show that G satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. Let y be such that $y = \lambda N y$ for some $\lambda \in (0,1)$. Thus

$$(Ny)(t) = \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^{m} W_k(t, y(t_k)).$$

From (H2), (A1) we have

$$\begin{aligned}
|y(t)| &= \lambda \left| \int_0^1 G(t,s)f(s,y(s))\Delta s + \sum_{k=1}^{m} W_k(t,y(t_k)) \right| \\
&\le \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 p(s)\psi(|y(s)|)\Delta s \\
&\quad +2\sum_{k=0}^{m}[c_k + \bar{c}_k] \\
&\le \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 p(s)\psi(\|y\|_{PC})\Delta s \\
&\quad +2\sum_{k=0}^{m}[c_k + \bar{c}_k].
\end{aligned}$$

Consequently,

$$\frac{\|y\|_{PC}}{\sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 p(s)\psi(\|y\|_{PC})\Delta s + 2\sum_{k=0}^{m}[c_k + \bar{c}_k]} \le 1$$

Then by (A1) there exists r such that $\|y\|_{PC} \neq r$.
Set

$$U = \{y \in C([0,T], \mathbb{R}) : \|y\|_{PC} < r\}.$$

As in Theorem 3.1 the operator $N : \overline{U} \to C([0,T)], \mathbb{R})$ is continuous and completely continuous. From the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, [18], we deduce that N has a fixed point y in U, which is a solution of the boundary value problem (1)–(4). □

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